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An upwind finite-volume element scheme and its maximum-principle-preserving property for nonlinear convection–diffusion problem

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SUMMARY

For a class of nonlinear convection–diffusion equation in multiple space dimensions, a kind of upwind finite-volume element (UFVE) scheme is put forward. Some techniques, such as calculus of variations, commutating operators and prior estimates, are adopted. It is proved that the UFVE scheme is unconditionally stable and satisfies maximum principle. Optimal-order estimates in H^1-norm are derived to determine the error in the approximate solution. Numerical results are presented to observe the performance of the scheme. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The finite-volume element (FVE) scheme is a discretization technique for partial differential equations, especially for those arising from physical conservation laws including mass, momentum and energy. This method has been introduced and analyzed by Li and his collaborators since the 1980s, see [1] for details. The FVE scheme uses a volume integral formulation of the original problem and a finite partitioning set of covolumes to discretize the equations. The approximate solution is chosen out of finite element spaces [1–3]. The FVE scheme is widely used in computational fluid mechanics and heat transfer problems [2–5]. It possesses the important and crucial property of

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inheriting the physical conservation laws of the original problem locally. Thus, it can be expected to capture shocks, to produce simple stencils or to study other physical phenomena more effectively.

On the other hand, The convection-dominated diffusion problem has strong hyperbolic characteristics, and therefore the numerical method is very difficult in mathematics and mechanics. When the central difference method, although it has second-order accuracy, is used to solve the convection-dominated diffusion problem, it produces numerical diffusion and oscillation, making numerical simulation a failure. The case usually occurs when the finite element methods (FEMs) and the FVE schemes are used for solving the convection-dominated diffusion problem.

For the two-phase plane incompressible displacement problem which is assumed to be Ω-periodic, Jim Douglas and Russell have published some articles on the characteristic finite difference method (FDM) and the FEM to solve the convection-dominated diffusion problems and to overcome oscillation and faults likely occurring in the traditional method [6]. Tabata and his collaborators have been studying upwind schemes-based triangulation for the convection–diffusion problem since 1977 [7–11]. Yuan Yirang, starting from the practical exploration and development of oil-gas resources, put forward an upwind finite difference fractional step method for the two-phase three-dimensional compressible displacement problem [12].

Most of the papers concern the FVE schemes for one- and two-dimensional linear partial differential equations [1–4, 13, 14]. In recent years, by introducing lumping operator, Feistauer *et al.* [15, 16] constructed several finite volume–FEMs for nonlinear convection–diffusion problems. On the other hand, because the FEMs cost great expense to solve the multiple space problems, we usually use the FDMs to approximate the problems [12]. These works led us to look into the subject of how to use the upwind finite volume element (UFVE) scheme to solve multiple space nonlinear convection-dominated diffusion problems.

In this article, we consider the following nonlinear convection–diffusion problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \mu\Delta u + \nabla\cdot\mathbf{F}(x,u) = g(x,u), && x\in\Omega, \quad t\in J=(0,T] \\ &u(x,t)=0, && x\in\Gamma, \quad t\in J \\ &u(x,0)=u_0(x), && x\in\Omega \end{aligned} \tag{1}$$

where $\Omega\subset R^3$ is a bounded region with boundary Γ, μ is a small positive constant number and $\mathbf{F}(x,u)=(f_1(x,u), f_2(x,u), f_3(x,u))^{\mathrm{T}}$ is a smooth vector function on $\bar{\Omega}\times R$, $\mathbf{F}(x,0)=\mathbf{0}$.

We put forward the UFVE scheme for solving the above multi-dimensional nonlinear convection-dominated diffusion problem. Some techniques, such as calculus of variations, commutating operator and prior error estimates, are adopted. We prove that the UFVE scheme is unconditionally stable and satisfies maximum principle. We also derive the optimal-order error estimates in H^1-norm. Numerical results show that the UFVE scheme is effective for avoiding numerical diffusion and nonphysical oscillations.

The remainder of this paper is organized as follows. In Section 2, we put forward the UFVE scheme for problem (1). In this section, we introduce some notations about mesh partition T_h and dual partition. The discrete maximum principle is derived in Section 3. Some auxiliary lemmas and the optimal-order error estimates in H^1-norm are proved in Section 4. In Section 5, numerical experiment shows that the method is effective for avoiding numerical diffusion and nonphysical oscillations.

Throughout this paper, the notations of standard Sobolev spaces $L^2(\Omega), H^k(\Omega)$ and associated norms $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}, \|\cdot\|_k = \|\cdot\|_{H^k(\Omega)}$ are adopted as those in [17, 18]. A constant C (without or with subscript) stands for a generic positive constant independent of discretization mesh parameter h, which may appear differently at different occurrences.

2. THE UFVE METHOD

We define a bounded set on $\mathbf{R}$:

$$G = u : |u| \leqslant K_0$$

where K_0 is a positive constant to be fixed later.

Suppose problem (1) satisfies condition (A1):

(C_1) *Continuity condition*: The function $g(x,u) \in L^2(\Omega \times R)$ is locally Lipschitz continuous with respect to the solution u. That is,

$$|g(x,u) - g(x,v)| \leqslant M|u-v| \quad \forall u, v \in G$$

(C_2) The function $\mathbf{F}(x,u)$ has first-order continuous partial derivative with respect to the variables x and u.

The exact solution u to problem (1) is smooth enough and satisfies the following regular condition:

(R) *Regular condition*: $u \in W^{2,\infty}(L^\infty) \cap H^1(L^2) \cap L^\infty(H^2)$.

Before presenting the numerical scheme, we introduce some notations. For simplicity, we assume that Ω is a cuboid domain $\Omega = (0, X_L) \times (0, Y_L) \times (0, Z_L)$. Firstly, let us consider a family of regular cuboid partition T_h in the domain $\bar{\Omega}$ (see [1]). Let h be the maximum diameter of cell of T_h. For a fixed cuboid partition $T_h = \{K\}$, we define a closed cuboid set $\{K_i\}_{i=1}^{N_K}$ and nodes set $\bar{\Omega}_h = \Omega_0 \cap \Gamma_h = \{P_i\}_{i=1}^{M_2}$, where $\Omega_0 = \{P_i\}_{i=1}^{M_1}$ is an inner nodes set of Ω, $\Gamma_h = \bar{\Omega} - \Omega_0 = \{P_i\}_{i=M_1+1}^{M_2}$ is a boundary nodes set on $\partial\Omega$. Let $E_h = \{e_i : 1 \leqslant i \leqslant M_E\}$ be an edges set.

Definition 2.1

Suppose that $T = \{T_h : 0 < h \leqslant h_0\}$ is a set of cuboid partition of Ω, the set T is called regular if there exists a positive constant σ_1 independent of h, such that

$$\max_{K \in T_h} \frac{h_K}{\rho_K} \leqslant \sigma_1 \quad \forall h \in (0, h_0)$$

where h_K and ρ_K are the diameters of K and the maximum diameter of circumscribing sphere of cuboid K, respectively.

Definition 2.2

The cuboid partition T_h is called Delaunnay mesh if K does not include the remainder of nodes of Ω_h for each $K \in T_h$.

Definition 2.3

The two cuboid cells are called face-adjacent if they have one common face and edge-adjacent if they have one common edge.

Definition 2.4

The two nodes are called adjacent if they form one edge which belongs to E_h. Denote $\bigwedge_i = \{j : P_j$ is adjacent to $P_i, P_i, P_j \in \Omega_h\}$.

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For a given cuboid partition T_h with nodes $\{P_i\}\in\Omega_h$ and edges $\{e_i\}\in E_h$, we construct two kinds of dual partitions. Firstly, we define the average center dual partition of T_h. $\forall P_i\in\Omega_h$, let $\Omega_h(P_i)=\{K:K\in T_h, P_i$ is a vertex of $K\}$. Let Q_j be a center of $K(\in\Omega_h(P_i)$. Connecting $Q_j(1\leqslant j\leqslant 6)$ of the two face-adjacent cuboid cell which belongs to $\Omega_h(P_i)$, we can derive a cuboid $K^*_{P_i}$ which surrounds the node P_i. $Q_j(1\leqslant j\leqslant 6)$ are vertexes of $K^*_{P_i}$ which is called average center dual partition corresponding to node P_i. $T^*_h=\{K^*_{P_i}:P_i\in\Omega_h\}$ is the average center dual partition of T_h. Suppose P_{ij} is the midpoint of P_i and its adjacent node P_j.

The other dual partition is defined as follows. $\forall e_k\in E_h$, let $\Omega_h(e_k)=\{K:K\in T_h$ and e_k be the edge of $K\}$. Let P_{k_1} and P_{k_2} be vertexes of the edge e_k and $Q_j(1\leqslant j\leqslant 4)$ be the center of the cuboid $K\in\Omega_h(e_k)$. Suppose that $K^*_{e_k}$ is a polyhedron whose vertexes are P_{k_1}, P_{k_2} and $Q_j(1\leqslant j\leqslant 4)$. $K^*_{e_k}$ is called a dual cell to the edge e_k. $\bar{T}^*_h=\{K^*_{e_k}\}_{k=1}^{M_E}$ is a dual partition to T_h.

Let Ω^*_h be the nodes set of dual partition. For $Q\in\Omega^*_h$, let K_Q be the cuboid cell which includes Q. Let $|K^*_P|$ and $|K_Q|$ be the volumes of the dual cells, K^*_P, and cuboid cell, K_Q, respectively. As what follows, we assume that the partition family T_h is regular, i.e. there exist positive constant C_1, C_2 independent of h, such that the following conditions (A2) are satisfied:

$$\begin{aligned} &C_1h^3\leqslant|K^*_p|\leqslant C_2h^3, \quad P\in\bar{\Omega}_h\\ &C_1h^3\leqslant|K_Q|\leqslant h^3, \quad Q\in\Omega^*_h \end{aligned}\tag{2}$$

Let the trial function space U_h be an isoparametric three-linear space based on T_h [1]. $U_h\subset H^1_0(\Omega)$, whose basis functions are $\{\varphi(Q)\}$, $Q\in\Omega^*_h$. Suppose that test function space $V_h(\subset L^2(\Omega))$ is a piecewise constant element space on dual partition T^*_h, whose basis functions are $\{\psi(P)\}$, $P\in\bar{\Omega}_h$ defined as follows:

$$\psi(P)=\begin{cases}1, & P\in K^*_P\\ 0 & \text{otherwise}\end{cases}$$

and $\psi(P)=0$, $P\in\Gamma_h$.

For the following analysis, suppose that $\Pi^*_h:H^1_0\to V_h$ is an interpolation operator, satisfying

$$\Pi^*_h u=\sum_{K^*_P\in T^*_h} u(P)\psi(P)\tag{3}$$

Multiplying both sides of (1) by v, then integrating on dual partition cell K^*_P, using the Green formulas and summing with respect to $P\in\bar{\Omega}_h$, we have

$$\left(\frac{\partial u}{\partial t},v\right)+a(u,v)+b(u,v)=(g(x,u),v),\quad v\in H^1_0(\Omega)\tag{4}$$

where

$$a(u,v)=\sum_{P\in\bar{\Omega}_h}\left[\mu\int_{K^*_P}\nabla u\cdot\nabla v\,\mathrm{d}x-\mu\int_{\partial K^*_P}\frac{\partial u}{\partial \nu}v\,\mathrm{d}s\right]\tag{5}$$

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$$b(u,v) = -\sum_{P\in\bar{\Omega}_h}\int_{K_P^*}\mathbf{F}(x,u)\cdot\nabla v\,\mathrm{d}x + \sum_{P\in\bar{\Omega}_h}\int_{\partial K_P^*}\mathbf{F}(x,u)\cdot v v\,\mathrm{d}s \tag{6}$$

We convert $\mathbf{F}$ to [1]

$$\mathbf{F}(x,u) = \int_0^u \frac{\partial \mathbf{F}(x,\bar{u})}{\partial \bar{u}}\,\mathrm{d}\bar{u} \tag{7}$$

Let

$$\begin{aligned} \beta_j^+(x,u) &= \int_0^u \max\left(0, \frac{\partial \mathbf{F}(x,\bar{u})}{\partial \bar{u}}\cdot v_j\right)\mathrm{d}\bar{u} \\ \beta_j^-(x,u) &= \int_0^u \max\left(0, -\frac{\partial \mathbf{F}(x,\bar{u})}{\partial \bar{u}}\cdot v_j\right)\mathrm{d}\bar{u} \end{aligned} \tag{8}$$

where $v_j (j=1,\ldots,6)$ are the unit outward normal vectors of $\Gamma_j \subset \partial K_P^*$. For $u_h \in U_h, v_h \in V_h$, we introduce a bilinear form as follows:

$$b_h(u_h, v_h) = \sum_{P\in\bar{\Omega}_h} v_h(P)\sum_{j=1}^{6}|\Gamma_j|\cdot[\beta_j^+(M_j, u_h(P)) - \beta_j^-(M_j, u_h(P_j))] \tag{9}$$

where $|\Gamma_j|$ is the area of Γ_j.

By far, we can obtain semidiscrete UFVE scheme: Find $u_h \in U_h$, such that

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + a(u_h, v_h) + b_h(u_h, v_h) = (g(x,u_h), v_h), \quad v_h \in V_h \tag{10}$$

Let $\Delta t = T/N$ and denote $t^n = n\Delta t, u^n = u(t^n), u_h^n = u_h(t^n), n = 1,2,\ldots,N$ and $\partial_t u_h^{n-1} = (u_h^n - u_h^{n-1})/\Delta t$. If the approximate solution $u_h^{n-1} \in U_h$ is known, then u_h^n can be found by the following semi-implicit full-discrete UFVE scheme:

$$(\Pi_h^*\partial_t u_h^{n-1}, v_h) + a(u_h^n, v_h) + b_h(u_h^{n-1}, v_h) = (g(x, u_h^{n-1}), v_h), \quad v_h \in V_h \tag{11}$$

3. DISCRETE MAXIMUM PRINCIPLE

For simplicity, we assume that the sides of all cuboid cells are parallel to coordinate axes, respectively. h_x, h_y, h_z denote isometric partition steps along X-, Y-, Z-directions. Let $h = \max\{h_x, h_y, h_z\}$.

Condition (A3): Suppose that the cuboid partition is regular, i.e. there exist positive constants C_3, C_4, such that

$$c_3 \leqslant \frac{h_x}{h_y}, \frac{h_x}{h_z}, \frac{h_y}{h_z} \leqslant c_4 \tag{12}$$

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By choosing $v_h = \psi(P)$ and dividing both sides of (11) by volume $|K_P^*| = h_x h_y h_z$ of the dual cell K_P^*, scheme (11) can be simplified as

$$\frac{1}{|K_P^*|}[(\partial_t u_h^{n-1}, \psi(P)) + a(u_h^n, \psi(P)) + b_h(u_h^{n-1}, \psi(P))]$$
$$= \frac{1}{|K_P^*|}(g(x, u_h^{n-1}), \psi(P)) \quad \forall P \in \Omega_0 \tag{13}$$

On the basis of cuboid mesh partition, define

$$F_j(x; u, v) = \beta_j^+(x, u) - \beta_j^-(x, v) \tag{14}$$

It is easy to prove that $F_j(x; u, v) : (R^3 \times R \times R) \to R \in C^0(R^3 \times R \times R)$ possesses the following properties:

(1) *Monotonicity*: $F_j(x; u, v)$ is monotonously nondecreasing with respect to the second variable and monotonously nonincreasing with respect to the third variable, i.e.

$$\partial_u F_j(x; u, v) \geqslant 0, \quad \partial_v F_j(x; u, v) \leqslant 0$$

(2) *Lipschitz continuity*: There exists a positive constant L_K, such that

$$|F_j(x_1; u_1, v_1) - F_j(x_2; u_2, v_2)| \leqslant h \cdot L_K(|x_1 - x_2| + |u_1 - u_2| + |v_1 - v_2|)$$

$\forall x_1, x_2 \in R^3, |u_i| < K, |v_i| < K, i = 1, 2, K > 0.$

(3) *Conservation*: $F_j(x; u, v) = -F_j(x; v, u), \forall x \in R^3, u, v \in R.$

For scheme (13), we have the following discrete maximum principle.

Theorem 3.1 (*Discrete maximum principle*)
Suppose that problem (1) satisfies condition (A1), and space partition steps satisfy conditions (A2), (A3). If time step Δt satisfies relation (A4):

$$\frac{6\Delta t h L_K}{|K_P^*|} \leqslant 1$$

then the approximate solution u_h^n of scheme (13) is bounded and satisfies

$$\|u_h\|_{L^\infty((0,T];L^\infty(\Omega))} \leqslant e^{CT}\|u^0\|_{L^\infty(\Omega)} + \tilde{C}T$$

where $\|u_h\|_{L^\infty((0,T];X)} = \max_{1\leqslant n\leqslant N}\|u_h^n\|_X$, $\tilde{C}$ is dependent on $g(x, u)$.

Proof
Firstly, combining (5)–(9) with (14), we can convert scheme (13) into

$$\left[1 + 2\mu\frac{\Delta t}{|K_P^*|}\left(\frac{h_x h_y}{h_z} + \frac{h_y h_z}{h_x} + \frac{h_z h_x}{h_y}\right)\right] u_h^n(P) - \mu\frac{\Delta t}{|K_P^*|}\left[\frac{h_y h_z}{h_x} u_h^n(P_1)\right.$$
$$\left. + \frac{h_z h_x}{h_y} u_h^n(P_2) + \frac{h_x h_y}{h_z} u_h^n(P_3) + \frac{h_y h_z}{h_x} u_h^n(P_4) + \frac{h_z h_x}{h_y} u_h^n(P_5) + \frac{h_x h_y}{h_z} u_h^n(P_6)\right]$$

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$$
\begin{aligned}
&= u_h^{n-1}(P) - \frac{\Delta t}{|K_P^*|} \sum_{j=1}^{6} |\Gamma_j| F_j(M_j; u_h^{n-1}(P), u_h^{n-1}(P_j)) \\
&\quad + \frac{\Delta t}{|K_P^*|} \int_{K_P^*} g(x, u_h^{n-1})\,\mathrm{d}x
\end{aligned} \tag{15}
$$

Here and hereafter, M_j $(j=1,\ldots,6)$ are midpoints of node P and nodes P_j $(j=1,\ldots,6)$ which are the adjacent nodes of node P. Let

$$
\tilde{F}_j^n = \begin{cases} \dfrac{F_j(P; u_h^n(P), u_h^n(P_j)) - F_j(P; u_h^n(P), u_h^n(P))}{u_h^n(P) - u_h^n(P_j)} & \text{if } u_h^n(P) \neq u_h^n(P_j) \\ 0 & \text{otherwise} \end{cases} \tag{16}
$$

From (15) and (16), we have

$$
\begin{aligned}
&\left[1 + 2\mu \frac{\Delta t}{|K_P^*|}\left(\frac{h_x h_y}{h_z} + \frac{h_y h_z}{h_x} + \frac{h_z h_x}{h_y}\right)\right] u_h^n(P) - \mu \frac{\Delta t}{|K_P^*|}\left[\frac{h_y h_z}{h_x} u_h^n(P_1)\right. \\
&\quad \left. + \frac{h_z h_x}{h_y} u_h^n(P_2) + \frac{h_x h_y}{h_z} u_h^n(P_3) + \frac{h_y h_z}{h_x} u_h^n(P_4) + \frac{h_z h_x}{h_y} u_h^n(P_5) + \frac{h_x h_y}{h_z} u_h^n(P_6)\right] \\
&= \left(1 - \frac{\Delta t}{|K_P^*|} \sum_{j=1}^{6} |\Gamma_j| \tilde{F}_j\right) u_h^{n-1}(P) + \frac{\Delta t}{|K_P^*|} \sum_{j=1}^{6} |\Gamma_j| \tilde{F}_j u_h^{n-1}(P_j) \\
&\quad - \frac{\Delta t}{|K_P^*|} \sum_{j=1}^{6} |\Gamma_j| [F_j(M_j; u_h^{n-1}(P), u_h^{n-1}(P_j)) - F_j(P; u_h^{n-1}(P), u_h^{n-1}(P_j))] \\
&\quad + \frac{\Delta t}{|K_P^*|} \int_{K_P^*} g(x, u_h^{n-1})\mathrm{d}x
\end{aligned} \tag{17}
$$

When all nodes in Ω_0 are chosen, the system of equations ($S1$) corresponding to scheme (13) for problem (1) is obtained. Obviously, the coefficient matrix of $S1$ is strictly diagonal dominance from (17). For the right-hand side of $S1$, making use of relation equation (A4), we have

$$
0 \leqslant \tilde{F}_j, \frac{\Delta t}{S_P^*} \sum_{j=1}^{6} |\Gamma_j| \tilde{F}_j \leqslant 1
$$

From relation equation (A4), condition (A2) and the property of $F_j(\cdot;\cdot,\cdot)$, we know that the third term of the right-hand side of $S1$ (RHS$S1$) is $O(\Delta t)$. Noting condition (A1), we know that the fourth term of RHS$S1$ is also $O(\Delta t)$. Hence, we have

$$
\max_{P \in \Omega_0} |u_h^n(P)| \leqslant \max_{P \in \Omega_0} |u_h^{n-1}(P)| + \tilde{C}\Delta t
$$

Note that $u_h^n(P) = 0, \forall P \in \Gamma_h$; we have

$$
\|u_h^n\|_{L^\infty(\Omega)} \leqslant (1 + C\Delta t)\|u_h^{n-1}\|_{L^\infty(\Omega)} + \tilde{C}\Delta t
$$

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By far, when $n=N$, we have

$$\begin{aligned}\|u_h^N\|_{L^\infty(\Omega)} &\leqslant (1+C\Delta t)^N\|u_h^0\|_{L^\infty(\Omega)}+\tilde{C}N\Delta t\\ &=(1+C\Delta t)^{T/\Delta t}\|u_h^0\|_{L^\infty(\Omega)}+\tilde{C}T\\ &=[(1+C\Delta t)^{1/(C\Delta t)}]^{CT}\|u_h^0\|_{L^\infty(\Omega)}+\tilde{C}T\\ &\leqslant e^{CT}\|u_h^0\|_{L^\infty(\Omega)}+\tilde{C}T\end{aligned}$$

In the above estimate, we use the fact that the relation inequality $(1+x)^{1/x}<e$ is true when x is sufficiently small. Hence, we have

$$\|u_h\|_{L^\infty((0,T];L^\infty(\Omega))}\leqslant e^{CT}\|u^0\|_{L^\infty(\Omega)}+\tilde{C}T \qquad \square$$

Remark
From Theorem 3.1, constant K_0 related to the set G can be fixed by

$$K_0=\max\{e^{CT}\|u^0\|_{L^\infty(\Omega)}+\tilde{C}T,\|u\|_{L^\infty(0,T;L^\infty)}\}$$

4. CONVERGENCE ANALYSIS

Define the discrete norm and the discrete semi-norm [1] as follows:

$$\|u_h\|_{0,h}^2=\|\Pi_h^* u_h\|_0^2=\sum_{K_{P_i}^*\in T_h^*}(u_h(P_i))^2|K_{P_i}^*| \tag{18}$$

$$|u_h|_{1,h}^2=\sum_{k=1}^{M_E}\left(\frac{u_h(P_{k2})-u_h(P_{k1})}{|e_k|}\right)^2|K_{e_k}^*| \tag{19}$$

$$\|u_h\|_{1,h}^2=\|u_h\|_{0,h}^2+|u_h|_{1,h}^2 \tag{20}$$

Obviously, the discrete norm and discrete semi-norm are equivalent with the corresponding continuous norms on U_h, respectively.

Lemma 4.1
Suppose that all cells K_Q of the partition T_h satisfy conditions (A2), (A3). T_h^* is circumcenter dual partition. $\forall u_h,\bar{u_h}\in U_h$, there exist positive constants γ, C_0 independent of h, such that

$$a(u_h,\Pi_h^* u_h)\geqslant\gamma\|u_h\|_1^2 \quad \forall u_h\in U_h \tag{21}$$

$$a(u_h,\Pi_h^*\bar{u}_h)\leqslant C_0\|u_h\|_1\|\bar{u}_h\|_1 \quad \forall u_h,\ \ \bar{u}_h\in U_h \tag{22}$$

$$|a(u_h,\Pi_h^*\bar{u_h})-a(\bar{u_h},\Pi_h^* u_h)|\leqslant Ch\|u_h\|_1\|\bar{u_h}\|_1 \tag{23}$$

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Proof
From the definition of $a(\cdot,\cdot)$ and the property of the function in U_h, we have

$$\begin{aligned}a(u_h,\Pi_h^* u_h) &= -\sum_{i=1}^{M_1} u_h(P_i)\int_{\partial K_{P_i}^*}\frac{\partial u_h}{\partial v}\,\mathrm{d}s\\ &= -\sum_{i=1}^{M_1} u_h(P_i)\sum_{j\in\bigwedge_i}\int_{\partial K_{P_i}^*\bigcap\partial K_{P_j}^*}\frac{u_h(P_j)-u_h(P_i)}{|P_iP_j|}\,\mathrm{d}s\\ &= \sum_{k=1}^{M_E}\int_{\partial K_{P_{k_1}}^*\bigcap\partial K_{P_{k_2}}^*}\frac{(u_h(P_{k_2})-u_h(P_{k_1}))^2}{|P_{k_1}P_{k_2}|}\,\mathrm{d}s\geqslant C|u_h|_{1,h}^2\end{aligned}$$

Noting the equivalence of $|\cdot|_{1,h}$ and $\|\cdot\|_1$, we can complete the proof of inequality (21) in Lemma 4.1. Analogously, we can derive the proof of inequality (22) and (23). □

Remarks
(i) From Lemma 4.1, we can say that $a(\cdot,\cdot)$ is positive definite in U_h.
(ii) Let $|\|u_h\||_1=[a(u_h,\Pi_h^* u_h)]^{1/2}$, then $|\|\cdot\||_1$ is equivalent to $\|\cdot\|_1$ in U_h.

Lemma 4.2
Let $|\|u_h\||_0=(\Pi_h^* u_h,\Pi_h^* u_h)^{1/2}$, $|\|\cdot\||_0$ is equivalent to $\|\cdot\|_0$ in U_h.
The proof of lemma 4.2 can be completed by computing integral on cell K_Q, directly .

Theorem 4.1 (*Trace Theorem* [*19*])
Suppose that Ω has a Lipschitz boundary, and that p is a real number in $[1,\infty]$; then there exists a constant C, such that

$$\|v\|_{L^p(\partial\Omega)}\leqslant C\|v\|_{L^p(\Omega)}^{1-1/p}\|v\|_{W_p^1(\Omega)}^{1/p}\quad \forall v\in W_p^1(\Omega)$$

Lemma 4.3
Suppose that P' is a random point in dual partition cell $K_{P_i}^*$, $\Gamma_{ij}=K_{P_i}^*\bigcap K_{P_j}^*$; then

$$\sum_{j\in\bigwedge_i}\int\int_{\Gamma_{ij}}|u(P')-u(x)|\,\mathrm{d}s\leqslant Ch^2(\|u\|_{1,K_{P_i}^*}+\|u\|_{2,K_{P_i}^*})\tag{24}$$

Proof
From Hölder's inequality, we obtain that

$$\sum_{j\in\bigwedge_i}\int\int_{\Gamma_{ij}}|u(P')-u(x)|\,\mathrm{d}s\leqslant Ch\sum_{j\in\bigwedge_i}(\int\int_{\Gamma_{ij}}|u(P')-u(x)|^2\,\mathrm{d}s)^{1/2}$$

Using Taylor's expansion, Hölder's inequality and trace theorem with $p=2$, the proof of Lemma 4.3 can be completed. □

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Lemma 4.4
For $\forall u_h, \bar{u_h} \in U_h$, there exists a positive constant C, such that

$$(u_h, \Pi_h^* \bar{u_h}) = (\bar{u_h}, \Pi_h^* u_h) \tag{25}$$

$$(u_h, \Pi_h^* \bar{u_h}) \leqslant C \|u_h\|_0 \cdot \|\bar{u_h}\|_0 \tag{26}$$

Lemma 4.5
Let $P_h u$ be the auxiliary elliptic projection of u; then we have

$$\|u - P_h u\|_1 \leqslant Ch\|u\|_2$$

We can complete the proofs of Lemmas 4.4 and 4.5 by calculating the integration directly.

Now, we turn to consider the error estimates of the approximate solution. Let

$$u^n - u_h^n = (u^n - P_h u^n) + (P_h u^n - u_h^n) = \rho_h^n + e_h^n$$

Choosing $t = t^n$ in (4), we have

$$\left(\frac{\partial u}{\partial t}(t^n), v\right) + a(u^n, v) + b(u^n, v) = (g(x, u^n), v) \quad v \in V_h \tag{27}$$

Subtracting (11) from (27), we obtain the error equation as follows:

$$\begin{aligned}(\Pi_h^* \partial_t e_h^{n-1}, v_h) + a(e_h^n, v_h) &= (r^n, v_h) + (b_h(u_h^{n-1}, v_h) - b(u^n, v_h)) \\ &\quad + (g(x, u^n) - g(x, u_h^{n-1}), v_h)\end{aligned} \tag{28}$$

where $r^n = \Pi_h^* \partial_t P_h u^{n-1} - u_t(t^n)$.

Choosing $v_h = \Pi_h^* \partial_t e_h^{n-1}$ in Equation (28), denoting by W_1, W_2 and T_1, T_2, T_3 the left- and right-hand side terms of Equation (28), we will analyze the five terms successively.

For W_1, making use of the discrete norm, the equivalence of $\|\cdot\|_{0,h}$ and $\|\cdot\|_0$, and Lemma 4.2, we know that there exists a positive constant C^*, such that

$$W_1 = (\Pi_h^* \partial_t e_h^{n-1}, \Pi_h^* \partial_t e_h^{n-1}) = \|\partial_t e_h^{n-1}\|_{0,h}^2 \geqslant C^* \|\partial_t e_h^{n-1}\| \tag{29}$$

For W_2, we have

$$\begin{aligned}W_2 &= a(e_h^n, \Pi_h^* \partial_t e_h^{n-1}) = a\left(e_h^n, \Pi_h^* \frac{e_h^n - e_h^{n-1}}{\Delta t}\right) \\ &= \frac{1}{2\Delta t}[a(e_h^n + e_h^{n-1}, \Pi_h^*(e_h^n - e_h^{n-1})) + a(e_h^n - e_h^{n-1}, \Pi_h^*(e_h^n - e_h^{n-1}))] \\ &= W_{21} + W_{22}\end{aligned} \tag{30}$$

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From (21) of Lemma 4.1, we can obtain the estimate of W_{22} as follows:

$$|W_{22}| \geqslant \frac{\gamma}{2\Delta t}\|e_h^n - e_h^{n-1}\|_1^2 \tag{31}$$

For W_{21}, rescript W_{21} as

$$\begin{aligned}
W_{21} &= \frac{1}{2\Delta t} a(e_h^n + e_h^{n-1}, \Pi_h^*(e_h^n - e_h^{n-1})) \\
&= \frac{1}{2\Delta t}[a(e_h^n, \Pi_h^* e_h^n) - a(e_h^{n-1}, \Pi_h^* e_h^{n-1}) + a(e_h^{n-1}, \Pi_h^* e_h^n) - a(e_h^n, \Pi_h^* e_h^{n-1})] \\
&= \frac{1}{2\Delta t}[|\|e_h^n\||_1^2 - |\|e_h^{n-1}\||_1^2] + \frac{1}{2\Delta t}[a(e_h^{n-1}, \Pi_h^* e_h^n) - a(e_h^n, \Pi_h^* e_h^{n-1})] \\
&= \frac{1}{2\Delta t}[|\|e_h^n\||_1^2 - |\|e_h^{n-1}\||_1^2] + \frac{1}{2\Delta t}[a(e_h^n + e_h^{n-1}, \Pi_h^*(e_h^n - e_h^{n-1})) \\
&\quad - a(e_h^n - e_h^{n-1}, \Pi_h^*(e_h^n + e_h^{n-1}))] \\
&= \frac{1}{2\Delta t}[|\|e_h^n\||_1^2 - |\|e_h^{n-1}\||_1^2] + \frac{1}{2}\left[a\left(e_h^n + e_h^{n-1}, \Pi_h^* \frac{e_h^n - e_h^{n-1}}{\Delta t}\right)\right. \\
&\quad \left. - a\left(\frac{e_h^n - e_h^{n-1}}{\Delta t}, \Pi_h^*(e_h^n + e_h^{n-1})\right)\right] \\
&\geqslant \frac{1}{2\Delta t}[|\|e_h^n\||_1^2 - |\|e_h^{n-1}\||_1^2] - Ch\|e_h^n + e_h^{n-1}\|_1 \|\partial_t e_h^{n-1}\|_1
\end{aligned}$$

From the equivalence of $|\|\cdot\||_1$ and $\|\cdot\|_1$, triangle inequality, inverse estimate [17, 18], we have

$$|W_{21}| \geqslant \frac{1}{2\Delta t}[(1 - C_1\Delta t)|\|e_h^n\||_1^2 - (1 + C_1\Delta t)|\|e_h^{n-1}\||_1^2] - \frac{C_*}{2}\|\partial_t e_h^{n-1}\|_0^2 \tag{32}$$

Hence, for the LHS of (28), we have

$$W_1 + W_2 \geqslant \frac{1}{2\Delta t}[(1 - C_1\Delta t)|\|e_h^n\||_1^2 - (1 + C_1\Delta t)|\|e_h^{n-1}\||_1^2] + \frac{C_*}{2}\|\partial_t e_h^{n-1}\|_0^2 \tag{33}$$

Using Lemma 4.4 and ε-inequality, we obtain

$$|T_1| = |(r^n, \partial_t e_h^{n-1})| \leqslant C\|r^n\|_0^2 + \frac{C_*}{6}\|\partial_t e_h^{n-1}\|_0^2 \tag{34}$$

Now we introduce the induction hypothesis:

$$\sup_{0 \leqslant n \leqslant N-1} |u^n - u_h^n|_{0,\infty} \leqslant M, \quad (h, \Delta t) \to 0 \tag{35}$$

and then from triangle inequality, we can obtain that $\sup_{0 \leqslant n \leqslant N-1} |u_h^n|_{0,\infty}$ is bounded.

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From the locally Lipschitz property of $g(x,u)$ in condition (C_2), making use of induction hypothesis (35), triangle inequality, important inequality and Lemma 4.4, we have

$$\begin{aligned}|T_3| &= |(g(x,u^n)-g(x,u_h^{n-1}),\Pi_h^*\partial e_h^{n-1})| \leqslant CK\|u^n-u_h^{n-1}\|_0\cdot\|\Pi_h^*\partial e_h^{n-1}\|_0\\ &\leqslant C\|u^n-u_h^{n-1}\|_0^2+\frac{C_*}{6}\|\partial_t e_h^{n-1}\|_0^2\\ &\leqslant C[\Delta t^2\|\partial_t u^{n-1}\|_0^2+\|\rho_h^{n-1}\|_0^2+\|e_h^{n-1}\|_0^2]+\frac{C_*}{6}\|\partial_t e_h^{n-1}\|_0^2 \end{aligned}\tag{36}$$

For T_2, for simplicity, we still denote by v_h the test function and let $\bar{u}_h=\partial_t \mathrm{e}_h^{n-1}\in U_h$, i.e. $v_h=\Pi_h^*\bar{u}_h$; then

$$\begin{aligned}T_2 &= b_h(u_h^{n-1},v_h)-b(u^n,v_h)\\ &= \sum_{P\in\Omega_0}\bar{u}_h(P)\sum_{j=1}^{6}|\Gamma_j|[\beta_j^+(u_h^{n-1}(P))-\beta_j^-(u_h^{n-1}(P_j))]\\ &\quad -\sum_{P\in\Omega_0}\bar{u}_h(P)\sum_{j=1}^{6}\int_{\Gamma_j}\mathbf{F}(x,u^n(x))\cdot v_j\,\mathrm{d}s\\ &= \sum_{P\in\Omega_0}\bar{u}_h(P)\sum_{j=1}^{6}\int_{\Gamma_j}[(\beta_j^+(u_h^{n-1}(P))-\beta_j^-(u_h^{n-1}(P_j)))- \ mathbf F(x,u^n(x))\cdot v_j]\,\mathrm{d}s\\ &= \sum_{P\in\Omega_0}\bar{u}_h(P)\sum_{j=1}^{6}\int_{\Gamma_j}[(\beta_j^+(u_h^{n-1}(P))-\beta_j^-(u_h^{n-1}(P_j)))-\mathbf{F}(M_j,u^n(x))\cdot v_j\\ &\quad +\mathbf{F}(M_j,u^n(x))\cdot v_j-\mathbf{F}(x,u^n(x))\cdot v_j]\,\mathrm{d}s\\ &= \sum_{P\in\Omega_0}\bar{u}_h(P)\sum_{j=1}^{6}\int_{\Gamma_j}[(\beta_j^+(u_h^{n-1}(P))-\beta_j^-(u_h^{n-1}(P_j)))\\ &\quad -(\beta_j^+(u^n(x))-\beta_j^-(u^n(x)))+(\mathbf{F}(M_j,u^n(x))\cdot v_j-\mathbf{F}(x,u^n(x))\cdot v_j)]\,\mathrm{d}s\\ &= \sum_{P\in\Omega_0}\bar{u}_h(P)\sum_{j=1}^{6}\int_{\Gamma_j}[(\beta_j^+(u_h^{n-1}(P))-\beta_j^+(u^n(x))\\ &\quad -(\beta_j^-(u_h^{n-1}(P_j)))-\beta_j^-(u^n(x)))+(\mathbf{F}(M_j,u^n(x))\cdot v_j-\mathbf{F}(x,u^n(x))\cdot v_{jl})]\,\mathrm{d}s\\ &= \sum_{P\in\Omega_0}\bar{u}_h(P)\sum_{j=1}^{6}\int_{\Gamma_j}(\beta_j^+(u_h^{n-1}(P))-\beta_j^+(u^n(x)))\,\mathrm{d}s\\ &\quad -\sum_{P\in\Omega_0}\bar{u}_h(P)\sum_{j=1}^{6}\int_{\Gamma_j}(\beta_j^-(u_h^{n-1}(P_j))-\beta_j^-(u^n(x)))\,\mathrm{d}s\\ &\quad +\sum_{P\in\Omega_0}\bar{u}_h(P)\sum_{j=1}^{6}\int_{\Gamma_j}(\mathbf{F}(M_j,u^n(x))\cdot v_j-\mathbf{F}(x,u^n(x))\cdot v_j)\,\mathrm{d}s\\ &= T_{21}+T_{22}+T_{23}\end{aligned}$$

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Thus, the estimate to T_2 is actually divided into the estimates to T_{21}, T_{22}, T_{23}. Firstly, from (8) we know that

$$
\begin{aligned}
|T_{21}| &= \left| \sum_{P\in\Omega_0} \bar{u}_h(P) \sum_{j=1}^{6} \int_{\Gamma_j} \beta_j^+(u_h^{n-1}(P)) - \beta_j^+(u^n(x))\,\mathrm{d}s \right| \\
&= \left| \sum_{P\in\Omega_0} \bar{u}_h(P) \sum_{j=1}^{6} \int_{\Gamma_j} \left(\int_{u^n(x)}^{u_h^{n-1}(P)} \max\left(0, \frac{\partial \mathbf{F}(M_j,\bar{u})}{\partial \bar{u}} \cdot v_j\right) \mathrm{d}\bar{u} \right) \mathrm{d}s \right| \\
&\leqslant \sum_{P\in\Omega_0} |\bar{u}_h(P)| \sum_{j=1}^{6} \int_{\Gamma_j} \left\| \frac{\partial \mathbf{F}(M_j,\bar{u})}{\partial \bar{u}} \right\|_\infty |u_h^{n-1}(P) - u^n(x)|\,\mathrm{d}s \\
&\leqslant \sum_{P\in\Omega_0} |\bar{u}_h(P)| \sum_{j=1}^{6} \left\| \frac{\partial \mathbf{F}(M_j,\bar{u})}{\partial \bar{u}} \right\|_\infty \cdot \int_{\Gamma_j} |u_h^{n-1}(P) - P_h u^{n-1}(P) + P_h u^{n-1}(P) - u^{n-1}(P) \\
&\quad + u^{n-1}(P) - u^{n-1}(x) + u^{n-1}(x) - u^n(x)|\,\mathrm{d}s
\end{aligned}
$$

From condition (C_2), Trace Theorem 4.1 and Lemma 4.3, we have

$$
|T_{21}| \leqslant C \left\| \frac{\partial \mathbf{F}(P_{jl},\bar{u})}{\partial \bar{u}} \right\|_\infty (\|e_h^{n-1}\|_1 + \|\rho_h^{n-1}\|_1 + h\|u^{n-1}\|_1 + \Delta t \|\partial_t u^{n-1}\|_1) \cdot |\bar{u}_h|_0
$$

Noting $\bar{u}_h = \partial_t e_h^{n-1}$, making using of Sobolev space interpolation theorem, triangle inequality and important inequality, after complex computing, we can obtain

$$
|T_{21}| \leqslant C[\|e_h^{n-1}\|_1^2 + h^2(\|u^{n-1}\|_2^2 + \|u^{n-1}\|_1^2) + (\Delta t)^2 \|\partial_t u^{n-1}\|_1^2] + \frac{C_*}{18} \|\partial_t e_h^{n-1}\|_0^2 \tag{37}
$$

For T_{22}, analogously, we have

$$
|T_{22}| \leqslant C[\|e_h^{n-1}\|_1^2 + h^2(\|u^{n-1}\|_2^2 + \|u^{n-1}\|_1^2) + (\Delta t)^2 \|\partial_t u^{n-1}\|_1^2] + \frac{C_*}{18} \|\partial_t e_h^{n-1}\|_0^2 \tag{38}
$$

For T_{23}, observing the properties of vector function $\mathbf{F}(x,u)$ and condition (C_2), making using of Taylor's expansion and triangle inequality, we can obtain

$$
|T_{23}| \leqslant Ch^4 + \frac{C_*}{18} \|\partial_t e_h^{n-1}\|_0^2 \tag{39}
$$

Combining (37)–(39), we obtain

$$
|T_2| \leqslant C(\|e_h^{n-1}\|_1^2 + h^2(\|u^{n-1}\|_2^2 + \|u^{n-1}\|_1^2) + (\Delta t)^2 \|\partial_t u^{n-1}\|_1^2 + h^4) + \frac{C_*}{6} \|\partial_t e_h^{n-1}\|_0^2 \tag{40}
$$

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Combining (35), (36) and (40) and applying Sobolev's space embedding theorem, the RHS of (28) satisfies

$$\begin{aligned}\text{RHS} \leqslant\; & C(\|r^n\|_0^2+\|e_h^{n-1}\|_1^2+(\Delta t)^2\|\partial_t u^{n-1}\|_0^2+h^2(\|u^{n-1}\|_2^2+\|u^{n-1}\|_1^2+h^2))\\ & +\frac{C_*}{2}\|\partial_t e_h^{n-1}\|_0^2\end{aligned} \tag{41}$$

From (33) and (41) we have

$$\begin{aligned}&\frac{1}{2\Delta t}[(1-C_1\Delta t)|\|e_h^n\||_1^2-(1+C_1\Delta t)|\|e_h^{n-1}\||_1^2]\\ &\quad \leqslant C(\|r^n\|_0^2+\|e_h^{n-1}\|_1^2+(\Delta t)^2\|\partial_t u^{n-1}\|_0^2+h^2(\|u^{n-1}\|_2^2+\|u^{n-1}\|_1^2+h^2))\end{aligned}$$

Observing the equivalence of $|\|\cdot\||_1$ with $\|\cdot\|_1$, the above equation can be rewritten as

$$\begin{aligned}|\|e_h^n\||_1^2 \leqslant\; & (1+C\Delta t)|\|e_h^{n-1}\||_1^2+C\Delta t(\|r^n\|_0^2+(\Delta t)^2\|\partial_t u^{n-1}\|_0^2\\ & +h^2(\|u^{n-1}\|_2^2+\|u^{n-1}\|_1^2+h^2))\end{aligned} \tag{42}$$

Deducing recurrently, we can obtain

$$\begin{aligned}|\|e_h^N\||_1^2 \leqslant\; & (1+C\Delta t)^N|\|e_h^0\||_1^2+C\Delta t\sum_{l=1}^{N}\|r^l\|_0^2\\ & +C\Delta t\sum_{l=0}^{N-1}[(\Delta t)^2\|\partial_t u^l\|_0^2+h^2(\|u^l\|_2^2+\|u^l\|_1^2+h^2)]\end{aligned} \tag{43}$$

For r^l, we have

$$\begin{aligned} r^l &= \Pi_h^*\partial_t P_h u^{l-1}-u_t(t^l)\\ &= \Pi_h^*\partial_t P_h u^{l-1}-\partial_t P_h u^{l-1}+(\partial_t P_h u^{l-1}-\partial_t u^{l-1})+(\partial_t u^{l-1}-u_t(t_l))\\ &= \Pi_h^*\partial_t P_h u^{l-1}-\partial_t P_h u^{l-1}+\frac{1}{\Delta t}\left[\int_{t_{l-1}}^{t_l}(P_h u_t-u_t)\,\mathrm{d}t-\int_{t_{l-1}}^{t_l}(t_l-t)u_{tt}\,\mathrm{d}t\right]\end{aligned}$$

After calculating the integration directly, using interpolation theorem, we can obtain an estimate of r^l. Replacing the terms r^l in the former equation by their bounds, summing it on n, we have

$$\sum_{l=1}^{N}\|r^l\|_0^2\leqslant Ch^2\sum_{l=1}^{N}\|\partial_t P_h u^{l-1}\|_1^2+C\frac{h^2}{\Delta t}\int_0^{t_N}\|u_t\|_2^2\,\mathrm{d}t+C\Delta t\int_0^{t_N}\|u_{tt}\|_0^2\,\mathrm{d}t \tag{44}$$

Substituting (44) into (43), and noting that $(1+C\Delta t)^N \leqslant \mathrm{e}^{CT} \equiv C$, we can obtain

$$
\begin{aligned}
|||e_h^N|||_1^2 \leqslant C &\left\{ |||e_h^0|||_1^2 + h^2 \int_0^{t_N} \|u_t\|_2^2\,\mathrm{d}t + (\Delta t)^2 \int_0^{t_N} \|u_{tt}\|_0^2\,\mathrm{d}t \right. \\
&\left. + \Delta t \sum_{l=0}^{N-1} [(\Delta t)^2 \|\partial_t u^l\|_0^2 + h^2(\|u^l\|_2^2 + \|u^l\|_1^2 + h^2)] \right\} \\
= C\{ &|||e_h^0|||_1^2 + h^2(\|u_t\|^2_{L^2((0,T];H^2(\Omega))} + \|u\|_{\bar{L}_\infty((0,T],H^2(\Omega))} \\
&+ \|u\|_{\bar{L}_\infty((0,T],H^1(\Omega))} + h^2) + (\Delta t)^2(\|u_{tt}\|^2_{L^2((0,T];L^2(\Omega))} + \|u_t\|^2_{\bar{L}_\infty((0,T];L^2(\Omega))})\}
\end{aligned} \tag{45}
$$

Observing that $N\Delta t \leqslant T$, the equivalence of $|||\cdot|||_1$ and $\|\cdot\|_1$ and elliptic projection theorem, we have

$$
\|u-u_h\|_{\bar{L}_\infty((0,T],H_1(\Omega))} = O(h+\Delta t) \tag{46}
$$

where $\|v\|_{\bar{L}_\infty((0,T],X)} = \sup_{n\Delta t \leqslant T} \|v^n\|_X$.

It remains to check the induction hypothesis (35). Firstly, for $n=0$, noting that u_h^0 is the interpolation of u_0, (35) holds, obviously. Suppose that (35) holds for $1 \leqslant n \leqslant N-1$. By (46), inverse estimate [17, 18] and dissectible relation $\Delta t = O(h)$, we have

$$
\|u^N - u_h^N\|_{0,\infty} \leqslant C\left(1+\log\frac{1}{h}\right)^{2/3} h^{-1/2} \|u^N - u_h^N\|_1
$$

Noting that the dissectible relation $\Delta t = O(h)$ and (46), we have

$$
\|u^N - u_h^N\|_{0,\infty} \leqslant C\left(1+\log\frac{1}{h}\right)^{2/3} h^{-1/2} h \leqslant C\left(1+\log\frac{1}{h}\right)^{2/3} h^{1/2}
$$

Observing that $(1+\log\frac{1}{h})^{2/3} h^{1/2} \to 0$, as $h \to 0$, we can obtain that

$$
\|u^N - u_h^N\|_{0,\infty} \leqslant M
$$

That is to say, the induction hypothesis (35) holds for $n=N$. Therefore, we have the following theorem.

Theorem 4.2

Suppose that the solution of problem (1) is sufficiently smooth, h and Δt are sufficiently small, $\Delta t = O(h)$ and the initial value u_h^0 is chosen as interpolation of u_0, then the error estimate (46) holds.

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5. NUMERICAL EXPERIMENT

Example 1

First, we discuss a series of linear convection-dominated diffusion equations in three dimensions.

$$\begin{aligned} &\frac{\partial u}{\partial t} - \mu \Delta u + \nabla \cdot (\mathbf{b} u) = f, \quad (x, y, z) \in \Omega, \quad t \in (0, T] \\ &u|_{t=0} = x(1-x)y(1-y)z(1-z), \quad (x, y, z) \in \Omega \\ &u|_{\partial\Omega} = 0, \quad t \in (0, T] \end{aligned} \tag{47}$$

The exact solution is chosen to be $u = e^{t/4}x(1-x)y(1-y)z(1-z)$, $\mathbf{b} = (1,1,0)$ and $\Omega = (0,1) \times (0,1) \times (0,1)$. We choose the time step as $\Delta t = 0.01$. Numerical results at $t = 0.5$ are presented in Tables I–III, where $h = 1/m$, $\|\cdot\|_0$, Order denote the space mesh step, the discrete L^2-norm, the error convergence order, respectively. $E_{\max}$ is the maximum absolute error. Some of these will be defined in Example 2.

Example 2

Second, we test a nonlinear problem on the same domain as that in Example 1, with $\mu = 10^{-4}$ and 10^{-6}, respectively. We set $\mathbf{F}(\mathbf{x}, u) = (\frac{1}{2}u^2, \frac{9}{4}u - \frac{1}{2}u^2, \frac{1}{2}u^2)$, and choose the right-hand side $g(\mathbf{x}, u)$

Table I. The error table at $\mu = 10^{-3}$.

m	$\|u-u_h\|_0$	Order	$E_{\max}$
10	2.1284e−003		5.3403e−003
20	1.1599e−003	0.8758	2.8869e−003
30	7.9937e−004	0.9181	1.9712e−003

Table II. The error table at $\mu = 10^{-4}$.

m	$\|u-u_h\|_0$	Order	$E_{\max}$
10	2.1423e−003		5.3717e−003
20	1.1674e−003	0.8759	2.9061e−003
30	8.0017e−004	0.9315	1.9787e−003

Table III. The error table at $\mu = 10^{-5}$.

m	$\|u-u_h\|_0$	Order	$E_{\max}$
10	2.1437e−003		5.3749e−003
20	1.1681e−003	0.8759	2.9081e−003
30	8.0025e−004	0.9328	1.9796e−003

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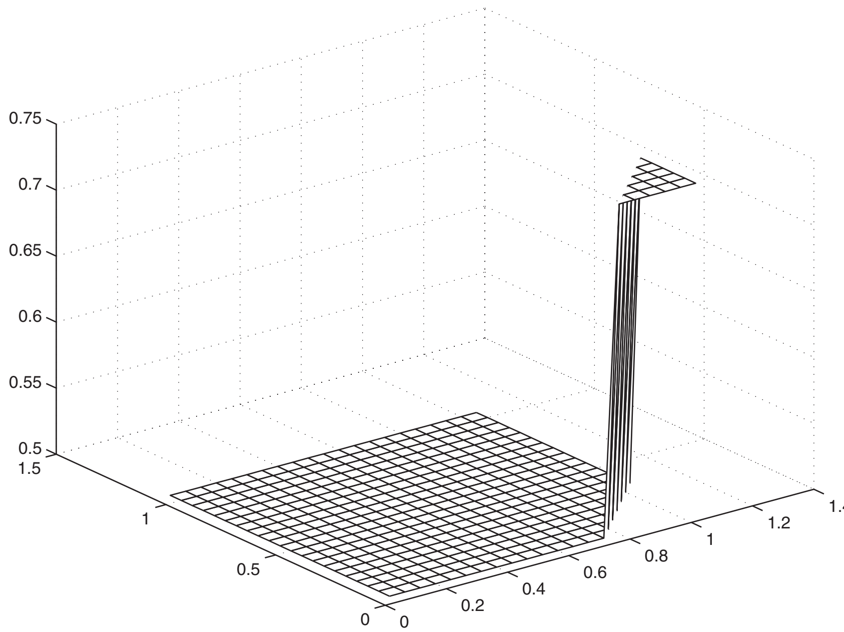

Figure 1. The exact solution at $t=0.5$ when $\mu=10^{-4}$.

in such a way that

$$u(x,y,z,t)=\frac{3}{4}-\frac{1}{4}\frac{1}{1+e^{(-4x+4y-4z-3/2t)/(32\mu)}} \tag{48}$$

is the exact solution. The initial condition and the boundary values are obtained directly from (48). The exact solution u and approximate solution u_h at $z=0.55$ are listed in Figures 1–4.

We set space step $h=0.05$, time step $\Delta t=0.01$ which satisfies dissectible relation. We define some norms as follows:

$$\|u-u_h\|_0=\left(\sum_{P\in\Omega_h}|u(P)-u_h(P)|^2|K_P^*|\right)^{1/2}$$

$$E_{\max}=\max_{P\in\Omega_h}|u(P)-u_h(P)|$$

$$RE_{\max}=\max_{P\in\Omega_h}\frac{|u(P)-u_h(P)|}{|u_h(P)|}$$

Numerical results at $t=0.5$ are presented in Table IV.

From figures and tables, we can see that our scheme is effective for avoiding numerical diffusion and nonphysical oscillations even though the problem has an abrupt slope solution.

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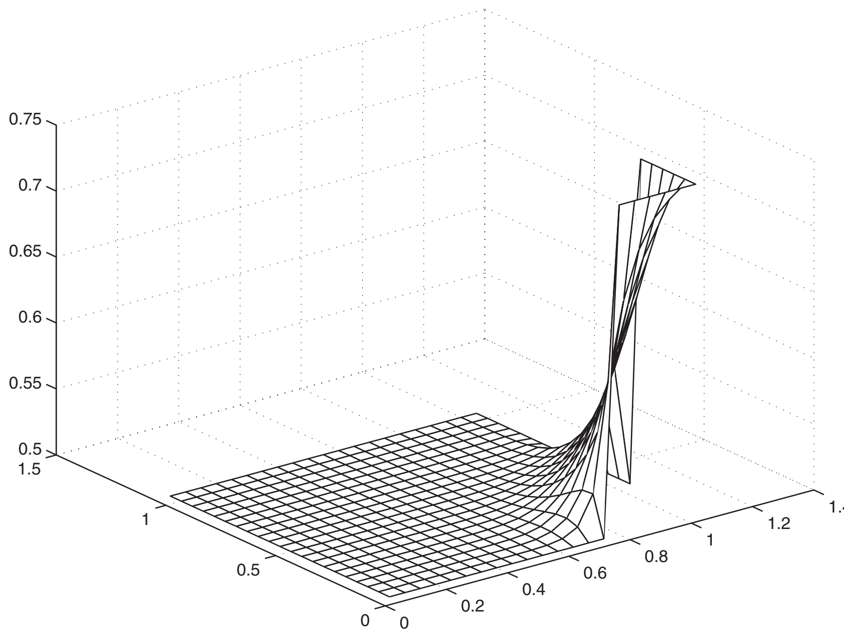

Figure 2. The approximate solution at $t=0.5$ when $\mu=10^{-4}$.

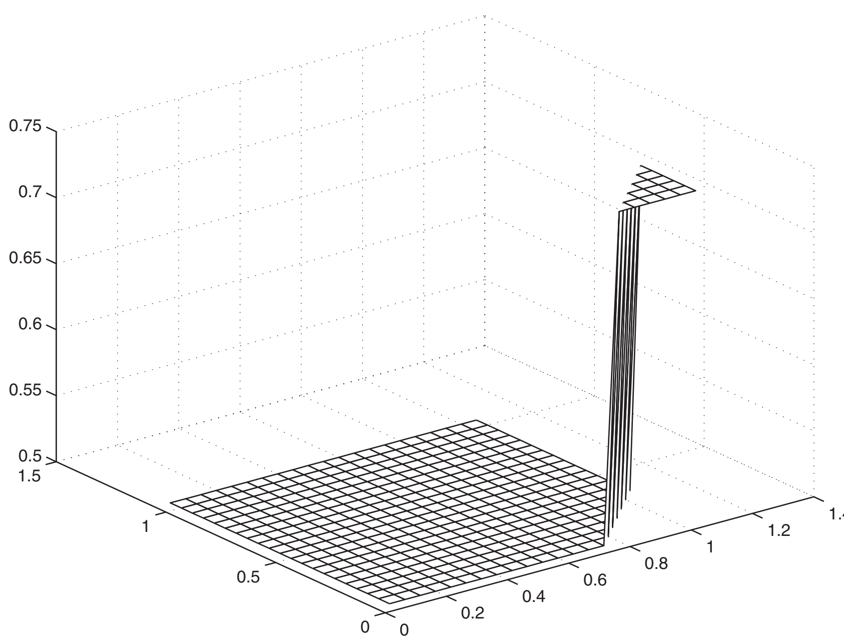

Figure 3. The exact solution at $t=0.5$ when $\mu=10^{-6}$.

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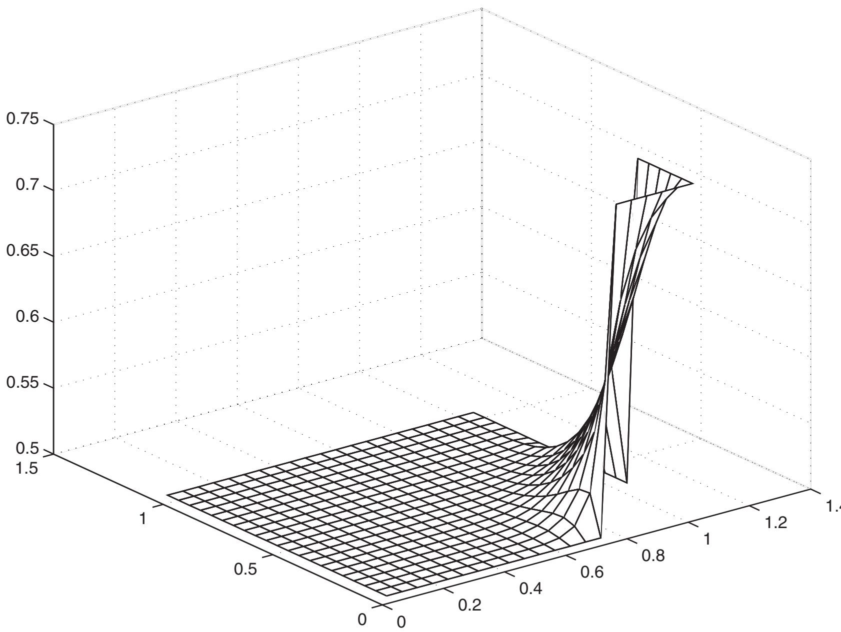

Figure 4. The approximate solution at $t = 0.5$ when $\mu = 10^{-6}$.

Table IV. The error table.

	$\|u - u_h\|_0$	$E_{\max}$	$RE_{\max}$
$\mu = 10^{-4}$	2.1657e−002	1.2201e−001	2.4402e−001
$\mu = 10^{-6}$	2.1630e−002	1.2199e−001	2.4397e−001

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